The stereographic projection P maps the upper half space to the upper half space, and more specifically it maps the sphere $S(r)$ of radius r and center $(0,0,r)$ which sits on the origin $(0,0,0)$, to the plane of points whose last coordinate is r. In other words, it maps the sphere $S(r)$ to the horizontal plane through its center. The equation of $S(r)$ is

$$\frac{x^2+y^2+z^2}{2z} = r.$$

Exercise. If we scale the sphere $S(r)$ from the origin by a scale factor $s > 0$, then we get $S(sr)$, that is

$$sS(r) = S(sr).$$

If (x,y,z) is a point on $S(r)$, then $P(x,y,z)$ is defined to be the point (X,Y,r) which is on the straight line joining $(0,0,0)$ to (x,y,z). From this description we find that

$$P(x,y,z) = (X,Y,Z) = \frac{x^2+y^2+z^2}{2z^2}(x,y,z).$$

We also notice that because P preserves direction, the quantity

$$\frac{x^2+y^2+z^2}{2z^2} = \frac{X^2+Y^2+Z^2}{2Z^2}$$

Is an invariant of P, and similarly for the reciprocal,

$$\frac{2z^2}{x^2+y^2+z^2} = \frac{2Z^2}{X^2+Y^2+Z^2}.$$

Thus the inverse mapping Q of P is given by

$$Q(X,Y,Z) = (x,y,z) = \frac{2Z^2}{X^2+Y^2+Z^2}(X,Y,Z).$$

So as to put this in context, we can consider the generalization of this situation, where we have a foliation of some region of space by surfaces

$$\tilde{S}(t) = t\,\tilde{S}(1)$$

as the parameter t varies over the positive real numbers. Suppose that $\tilde{S}(t)$ is given by

$$f(x,y,z) = t,$$

Where f is say a continuously differentiable function. In order for these to not intersect we require that $\tilde{S}(1)$ contains precisely one point on each straight line through the origin, and is preferably contained in the upper half space and a continuously differentiable graph over the plane $z = 0$, so we don't need to worry about topology.

Scaling by $1/t$ we get back from $\tilde{S}(t)$ to $\tilde{S}(1)$, that is $\tilde{S}(1)$ is the set of those $(x/t , y/t , z/t)$ with $f(x, y, z) = t$, which is also by renaming dummy variables, the set of those (x, y, z) with

$$f(tx, \quad ty , \quad tz) = t.$$

But since $\tilde{S}(1)$ is also just the set of those (x, y, z) with

$$f(x, y, z) = 1,$$

And the two conditions defining $\tilde{S}(1)$ must be the same, we see that one implies the other and so

$$f(tx, \quad ty , \quad tz) = t\, f(x, y, z).$$

This just says that f is homogeneous of degree 1.

We want to generalize the stereographic projection to this situation. We define the generalized stereographic projection $\widetilde{P}$ which turns the surface $\tilde{S}(t)$ into the plane of points of the form (X, Y, t), by just specifying that (X, Y, t) is on the straight line through (0,0,0) and (x, y, z). Explicitly,

$$(X,\ \ Y,\qquad t\quad)\ =\ \frac{t}{z}\,(x,y,z)\ =\ \frac{f(x,y,z)}{z}\,(x,y,z)\ .$$

Thus it makes sense to set

$$g(x,y,z)\ \ =\ \ \frac{f(x,y,z)}{z}=f\left(\frac{x}{z},\qquad \frac{y}{z}\,,\ \ 1\right).$$

So g is scale invariant equivalently homogeneous of degree 0, that is $g(sx,sy,sz)=g(x,y,z)$ for $s>0$, and $\tilde{S}(t)$ is given by $z\,g(x,y,z)\ =\ t$, and

$$\widetilde{P}\,(x,y,z)=(X,Y,Z)=\ g(x,y,z)(x,y,z).$$

There is a curious ambiguity of notation because (x,y,z) is used to denote both the input of the function g, and the coordinate vector itself. This is really unfortunate, and could be avoided with more refined notation for example fancy parentheses to indicate we are evaluating the function g at that particular input. As it stands, just writing $g(x,y,z)$ could refer to g evaluated at (x,y,z) or the function g multiplied by the vector (x,y,z). We dearly want to avoid this kind of ambiguity. Let's for now be really French-German instead of depending on the reader to use their discretion. Let's agree to use square brackets when we are evaluating a function at an input and see how that works out. Thus our equation becomes

$$\widetilde{P}\,[(x,y,z)] = (X,Y,Z) = \ g[(x,y,z)]\ \ (x,y,z).$$

Notice that the last coordinate is $Z = gz$, or if we are going to labor over details, $Z = g[(x,y,z)]z$. Although we have been going backwards and forwards on whether we want to start with (x,y,z) or Z, we actually started off with the demand that the point (x,y,z) is on the surface $\tilde{S}[Z]$. Then we wrote Z in terms of (x,y,z) to now assert that (x,y,z) is on the surface $\tilde{S}[gz]$ so that everything is written nicely in terms of an arbitrary point (x,y,z). We really feel like true mathematicians when we can start with a special case, then figure out the general case from that one. We probably noticed before, that instead of starting with the surfaces $\tilde{S}[t]$, we could have started with the single surface $\tilde{S}[1]$, and then defined $\tilde{S}[t] = t\,\tilde{S}(1)$. This would lead us to consider what happens if we take weird choices of $\tilde{S}[1]$. However, now we have another plausible starting point, which is to consider any scale invariant function $g[(tx,ty,tz)] = g[(x,y,z)]$ at least for all positive values t. Then we can look at the surface $\tilde{S}[1]$ which is the set of points (x, y, z) where $g[(x, y, z)]z$ =1, and build up a theory from that starting point. However, we get scale invariant functions by knowing where they are equal to 1. We could think instead of starting with a conical surface called $\tilde{C}[1]$ consisting of a curve of points and all scaling of them so that $\tilde{C}[t] = t\tilde{C}[1]$ for every positive real number t. This is not enough however to define g everywhere. Knowing the level sets of gz gives much more information than just knowing the level set of g, which is necessarily a cone. In the case of standard stereographic projection, $C[1]$ is the surface

$$\frac{x^2 + y^2 + z^2}{2z^2} = 1,$$

Which is just the cone $x^2 + y^2 = z^2$. It is perhaps interesting to note that $C[t]$ is the cone

$$x^2 + y^2 = (2t - 1)z^2.$$

It is obtained from $C[1]$ by scaling along the z direction. It is an exercise in analysis to figure out what kind of family of symmetries exist or do not exist in general. For now we are just satisfied that a single level surface of gz determines the function g, by homogeneity, and inside a level surface of gz are the flat level curves of z or equivalently level curves of g.

In any case, for the moment we can leave those as homework investigations for pure math REUs.

The inverse $\tilde{Q}$ of $\widetilde{P}$ is given by

$$\widetilde{Q}\,[(X,Y,Z)] = (x,y,z) = \frac{1}{g[(x,y,z)]}(X,Y,Z)$$
$$= \frac{1}{g[(X,Y,Z)]}(X,Y,Z).$$

Our project now is to compute the first derivative of the mapping $\tilde{P}$, and thus see how $\tilde{P}$ behaves locally since the first derivative gives us the linear approximation. Let's then differentiate the equation

$$\widetilde{P}\,[(x,y,z)] = (X,Y,Z) = g[(x,y,z)](x,y,z).$$

To get a good look at the derivative, let's write our coordinates $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\begin{pmatrix} X \\ Y \\ Z \end{pmatrix}$ as column vectors

We have

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = g \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

and we can compute the matrix of partial derivatives

$$D\,\tilde{P} = \begin{pmatrix} \partial_x X & \partial_y X & \partial_z X \\ \partial_x Y & \partial_y Y & \partial_z Y \\ \partial_x Z & \partial_y Z & \partial_z Y \end{pmatrix}.$$

the linear approximation to $\widetilde{P}$ is

$$\widetilde{P} \begin{pmatrix} x + \delta x \\ y + \delta y \\ z + \delta z \end{pmatrix} \approx \widetilde{P} \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right] + D\,\tilde{P} \left[\begin{pmatrix} x \\ y \\ z \end{pmatrix} \right] \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

Where the error tends to zero faster than the vector $\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$. In other words, its size is $o\left(\sqrt{\delta x^2 + \delta y^2 + \delta z^2}\right)$.

For this purpose, because we are fundamentally interested in the coordinate $Z = gz$, we start by using the decomposition

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = gz \begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix}$$

Using the product rule, we obtain the matrix

$$D\,\tilde{P} = \begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix} \big(\partial_x\,(gz)\ \ \partial_y\,(gz)\ \ \partial_z\,(gz)\big)$$
$$+\ gz \begin{pmatrix} 1/z & 0 & -x/z^2 \\ 0 & 1/z & -y/z^2 \\ 0 & 0 & 0 \end{pmatrix}$$

Where the first term is defined by matrix multiplication. Let's pull out a factors $1/z$ again to write this as

$$\frac{1}{z}\begin{pmatrix} x \\ y \\ z \end{pmatrix} \big(\partial_x\,(gz)\ \ \partial_y\,(gz)\ \ \partial_z\,(gz)\big)\ +\ g \begin{pmatrix} 1 & 0 & -x/z \\ 0 & 1 & -y/z \\ 0 & 0 & 0 \end{pmatrix}$$

Right-multiplying this by $\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$, we get

$$D\,\widetilde{P}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$
$$= \begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix}\left(\partial_x\,(gz)\;\; \partial_y\,(gz)\;\; \partial_z\,(gz)\right)\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$
$$+\, g\begin{pmatrix} 1 & 0 & -x/z \\ 0 & 1 & -y/z \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}.$$

The first term can if we prefer be written in terms of the gradient $\nabla(gz)$ of gz and the dot product, instead of matrix multiplication.

$$\begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix} \quad \begin{pmatrix} \partial_x\,(gz) \\ \partial_y\,(gz) \\ \partial_z\,(gz) \end{pmatrix}\cdot\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$$

The gradient of a function is always perpendicular to the level sets of the function, so this is going to vanish precisely when $\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$ is tangent to the surface $\tilde{S}[gz]$, .

The map $\widetilde{P}$ is said to be *conformal* if the matrix $D\,\widetilde{P}$ is orthogonal. We can ask whether this ever happens for any choice of g. Write

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \nabla(gz) = \begin{pmatrix} \partial_x\,(gz) \\ \partial_y\,(gz) \\ \partial_z\,(gz) \end{pmatrix}.$$

Then $D\,\widetilde{P}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$

$$= \begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \cdot \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} + g\begin{pmatrix} 1 & 0 & -x/z \\ 0 & 1 & -y/z \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}.$$

Knowing that $\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$ is tangent to $\tilde{S}[gz]$ if and only if

$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \cdot \begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix} = 0,$ we see that $D\,\widetilde{P}\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$ has vanishing third component if $\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$ is in the tangent plane of $\tilde{S}[gz]$.

From this it is obvious already that if

We are going to compute this for three choices of the vector $\begin{pmatrix} \delta x \\ \delta y \\ \delta z \end{pmatrix}$, namely

$$\begin{pmatrix} -V_y \\ V_x \\ 0 \end{pmatrix}, \quad \begin{pmatrix} V_x \\ V_y \\ -(V_x^2 + V_y^2)/V_z \end{pmatrix}, \quad \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}.$$

We notice that these are pairwise orthogonal. The first two vectors evaluated at the point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ span the tangent plane of the surface $\tilde{S}[gz]$ at $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, and the last vector $\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$ is the normal.

We compute

$$D\,\widetilde{P}\begin{pmatrix} -V_y \\ V_x \\ 0 \end{pmatrix} = g\begin{pmatrix} -V_y \\ V_x \\ 0 \end{pmatrix}$$

Moreover,

$$D\,\widetilde{P}\begin{pmatrix} V_x \\ V_y \\ -\dfrac{V_x^2+V_y^2}{V\quad{}_z} \end{pmatrix} = g\begin{pmatrix} V_x + \dfrac{x}{z}\,\dfrac{\left(V_x^2+V_y^2\right)}{V_z} \\ V_y + \dfrac{y}{z}\,\dfrac{\left(V_x^2+V_y^2\right)}{V_z} \\ 0 \end{pmatrix}$$

$$= g\begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} + g\,\frac{\left(V_x^2+V_y^2\right)}{z\;V_z}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}.$$

We want to know the conditions on g which will ensure that $D\,\widetilde{P}$ maps the first two vectors to orthogonal vectors. This happens when

$$\begin{pmatrix} -V_y \\ V_x \\ 0 \end{pmatrix}\cdot\left(\begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} + \frac{\left(V_x^2+V_y^2\right)}{z\;V_z}\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}\right) = 0,$$

This boils down to

$$\frac{\left(V_x^2+V_y^2\right)}{V_z}\left(-x\,V_y + \;y\,V_x\,\right) \;= 0.$$

In other words, if V_z and at least one of V_x, V_y are non-zero, then $xV_y = y\,V_x$. Away from the planes where x, y vanish, there is a function W such that

$$V_x = z\,W, \quad V_y = y\,W.$$

Now $D\,\widetilde{P}$ scales the length of $\begin{pmatrix} -V_y \\ V_x \\ 0 \end{pmatrix}$ by $|g|$. If it is to be orthogonal on the tangent plane to the surface $\tilde{S}[gz]$, then it must scale the vector $\begin{pmatrix} V_x \\ V_y \\ -(V_x^2 + V_y^2)/V_z \end{pmatrix}$ by $|g|$ as well. Thus the two vectors

$$\begin{pmatrix} V_x \\ V_y \\ -(V_x^2 + V_y^2)/V_z \end{pmatrix}, \qquad \begin{pmatrix} V_x \\ V_y \\ 0 \end{pmatrix} + \frac{(V_x^2 + V_y^2)}{z\ V_z} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Must have the same length. Rewriting these in terms of W, and writing we see that

$$\begin{pmatrix} x \\ y \\ -\dfrac{(x^2 + y^2)}{V_z} \end{pmatrix}, \quad \left(1 + \frac{W(x^2 + y^2)}{z\ V_z}\right) \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

Must have the same length. This gives

$$x^2 + y^2 + \frac{(x^2+y^2)^2}{V_z^2}$$

$$= \ (x^2+y^2)\left(1 + \frac{W(x^2+y^2)}{z\ V_z}\right)^2.$$

This immediately simplifies to

$$1 + \frac{(x^2+y^2)}{V_z^2} \ = \ \left(1 + \frac{W(x^2+y^2)}{z\ V_z}\right)^2.$$

Then to

$$\frac{(x^2+y^2)}{V_z^2} \ = \ \frac{2W(x^2+y^2)}{z\ V_z} \ + \ \frac{W^2(x^2+y^2)^2}{z^2\ V_z^2} \ .$$

Clearing denominators gives

$$(x^2+y^2)z^2 \ = \ 2W(x^2+y^2)zV_z \ + W^2(x^2+y^2)^2$$

Hence

$$V_z \ = \frac{z}{2W} - \ \frac{W(x^2+y^2)}{2z}$$

We have found the formulas which make $\tilde{P}$ a conformal transformation when restricted to each of the surfaces $\tilde{S}\ (r)$, namely

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = \begin{pmatrix} xW \\ yW \\ \frac{z}{2W} - \frac{W(x^2+y^2)}{2z} \end{pmatrix}.$$

$$\nabla (gz) = \begin{pmatrix} xW \\ yW \\ \frac{z}{2W} - \frac{W(x^2+y^2)}{2z} \end{pmatrix}. \quad (1)$$

Let’s check the case of the stereographic projection

$$gz\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \frac{x^2+y^2+z^2}{2z}.$$

We see that

$$\nabla (gz) = \begin{pmatrix} x/z \\ y/z \\ \frac{1}{2} - \frac{(x^2+y^2)}{2z^2} \end{pmatrix}.$$

Hence we set $w = 1/z$, and this works perfectly. We are interested more generally in the choices of W for which the equation (1) has a solution.

We notice that we can integrate the equation

$$\nabla_x (gz) = x/z$$

Now

$$\partial_x(gz) = xW, \qquad \partial_y(gz) = yW.$$

Hence

$$\left(\mathrm{y}\ \partial_x\ - x\ \ \partial_y\right)(gz) = \ \ 0.$$

However, changing to cylindrical polars, we have that

$$x = r\cos\theta\ , \qquad y = r\sin\theta,$$

$$\partial_r \ = \ \cos\theta\ \ \partial_x\ + \ \ \sin\theta\ \ \partial_y$$

$$\partial_\theta \ = \ -\mathrm{r}\sin\theta\ \ \partial_x\ + \ \ r\ \cos\theta\ \ \partial_y.$$

Which we can write

$$\begin{pmatrix}\partial_r\\ \partial_\theta\end{pmatrix} \ = \ \begin{pmatrix}\cos\theta & \sin\theta\\ -r\sin\theta & r\cos\theta\end{pmatrix}\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix}$$

And inversely,

$$\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} \ = \ \begin{pmatrix}\cos\theta & \dfrac{-\sin\theta}{r}\\ \sin\theta & \dfrac{\cos\theta}{r}\end{pmatrix}\begin{pmatrix}\partial_r\\ \partial_\theta\end{pmatrix}$$

$$\left(\mathrm{y}\ \partial_x\ - x\ \ \partial_y\right) = \ - \partial_\theta.$$

We discover that gz is independent of the angle θ. Thus $gz = f[(r,z)]$, for some function f. Now

$$\left(\mathrm{x}\ \partial_x + \ y\ \ \partial_y\right)(gz)\ = (x^2 + \ y^2)W.$$

Since

$$\mathrm{x}\ \partial_x + \ y\ \ \partial_y\ =\ \ r\ \partial_r$$

We can write the equation as

$$r\ \partial_r\, f =\ \ \ r^2\, W.$$

So in particular W is also independent of the angle θ. After this change of variables,

$$\partial_r\, f\ \ =\ \ r\, W[(r,z)].$$

We know however, that W must be homogeneous of degree -1. It is obtained by dividing components of the scale invariant function $\nabla(gz)$ by coordinate functions which are homogeneous of degree 1. Indeed,

$$W[(tr, tz)] = \frac{W[(r,z)]}{t}.$$

Let's leave this as an exercise for later, and go the last mile, and see whether the mapping $\tilde{P}$ is ever conformal on a region of space. For this we need to compute

$$D\,\widetilde{P}\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$$

$$= \left(V_x^2 + V_y^2 + V_z^2\right)\begin{pmatrix} x/z \\ y/z \\ 1 \end{pmatrix}$$

$$+ g\begin{pmatrix} V_x - \left(\frac{x}{z}\right)V_z \\ V_y - \left(\frac{y}{z}\right)V_z \\ 0 \end{pmatrix}.$$

Let's plug in our solutions for $V_x,\ V_y,\ V_z$.

Let’s look at the well known case of the stereographic projection.

$$gz \quad = \quad \frac{x^2 + y^2 + z^2}{2z}$$

$$\begin{pmatrix} \partial_x \,(gz) \\ \partial_y \,(gz) \\ \partial_z \,(gz) \end{pmatrix} \quad = \begin{pmatrix} x/z \\ y/z \\ \frac{1}{2} - \frac{x^2 + y^2}{2z^2} \end{pmatrix}.$$

We compute

$$V_x^2 + V_y^2 + V_z^2$$

$$= \quad \frac{x^2 + y^2}{z^2} + \frac{1}{4} - \frac{x^2 + y^2}{2z^2} + \left(\frac{x^2 + y^2}{2z^2}\right)^2.$$

This is just

$$\left(\frac{x^2 + y^2 + z^2}{2z^2}\right)^2.$$

Putting the terms together yields

$$\left(\frac{x^2 + y^2 + z^2}{2z^2}\right)^2 \begin{pmatrix} 2x/z \\ 2y/z \\ 1 \end{pmatrix}$$

We really need to stare at this calculation and understand it better. It is very upsetting that whilst a single change in sign would lead to this vector being in the direction $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and the mapping $\tilde{P}$ being conformal on space as well as when restricted to the sphere $S(r)$, the signs just do not seem to be working out today. It might be that we need to complexify something or it might be that we made a mistake or it might just be a fact of life that this is tantalizingly close but no cigar.