Kate Okikiolu's Lecuture Notes Assignment, 1/25/2026: Write more words on the Three Lines Lemma from Terry Tao's lecture notes. The lemma itself has been taught for generations and appears in Terry's notes.

Lemma 5.10 (Three Lines Lemma.) Let f be a complex-analytic function on the strip $\{0 \leq \mathrm{Re}(z) \leq 1\}$, which is of at most double-exponential growth, or more precisely

$$|f(z)| \leq C \exp\Big(\quad C_0 \ \exp\big((\pi - \delta)|z|\big) \ \Big)$$

for some $\delta > 0$. Suppose that we have the bounds $|f(z)| \leq A$ when $\mathrm{Re}\, z = 0$ and $|f(z)| \leq B$ when 1.

Then we have

$$|f(z)| \leq A^{1-\mathrm{Re}(z)} \ B^{\mathrm{Re}(z)}$$

Discussion. Before discussing this result, let's pose a homework question to think about during our discussion. The strip has two ends. What can we say if g just satisfies the conditions of the theorem on one end?

The first thing to note about the lemma is that the domain of the analytic function is an unbounded strip in the complex plane. Thus the famous maximum modulus principle for complex analytic functions does not apply. That result tells us that if a complex function is defined on a bounded open set in the complex plane and if the function is complex-analytic, and moreover extends continuously to the boundary of the set, then the function attains its maximum modulus on the boundary. We would like to apply that theorem here. However there are two problems and the first one is obviously that the region in question is not bounded.

If U is an unbounded open set in the complex plane, and if g is a complex-analytic function on U which extends continuously to the boundary, we say that g is unbounded but anchored, which we abbreviate to UBA, if g is uniformly bounded on the boundary of U, but unbounded on U. The Lemma is telling us that UBA functions on the strip of width 1 in the complex plane have to grow faster than $\exp\big(\exp\big((\pi - \delta)|z|\big)\big)$.

The Lemma is not just a trivial application of the maximum modulus principle. We need to work on the function f. If we just try to apply the maximum modulus principle to f on the rectangles

$$\{0 \leq \mathrm{Re}(z) \leq 1\}, \qquad \{-N \leq \mathrm{Im}(z) \leq N\},$$

where N is large we get nothing. The bounds we assume for the modulus $|f(z)|$ on the lines $|\mathrm{Im}(z)| = N$ increase to infinity as N increases, with a horrible double exponential asymptotic. Secondly we want a nice delicate refined bound on $|f|$ within the strip that is better than just taking the maximum of A and B.

The proof of the result splits into two cases, most importantly, the case when $A = B = 1$, and more easily the problem of getting to the general case from that case. These two cases both have guiding example functions. We cannot hope to understand the lemma without understanding these examples. The easier part of the result is reducing the general case to the case $A = B = 1$. Let's assume that A, B are two positive numbers. (The reader can deal with the case when one of them vanishes at the end.) The guiding example is the function

$$A\exp(cz).$$

Here, c is some real valued constant which is not zero. Writing $z = x + iy$ with x, y being its real and imaginary parts, we have

$$A\exp(cz) \ = \ A\exp(cx + ic\,y) = A\exp(cx)\exp(icy).$$

Of course the last factor is just a complex number of modulus 1.

$$\exp(icy) \ = \ \cos(cy) + \ i \ \sin(cy).$$

The modulus of $A\exp(cz)$ is precisely $A\exp(cx)$. It is constant on each line $x + iy$ of the strip with x fixed. Of course $A\exp(cz)$ is not constant on this line because it winds around and around on the complex circle of numbers with modulus $A\exp(cx)$. The lemma is however a result about the modulus of analytic functions on the strip, and so this is irrelevant. In this guiding example, the argument of the function changes but its modulus is bounded on the whole strip. What is now important for us is that whatever positive value B

we are given, we can select c so that $A \exp c = B$. Indeed, just take $c = \log(B/A)$. We then notice that

$$A \exp(cx) = A \exp(x \log(B/A)) = A \left(\frac{B}{A}\right)^x$$
$$= A^{1-x} B^x.$$

This is the beautiful precise bound in the lemma. Our guiding example is equal to $A^{1-z} B^z$.

If we start with a general function f satisfying the hypotheses of the lemma with general positive constants A, B, then the reader can check that the new function

$$\frac{f(z)}{A \exp(cz)} = \frac{f(z)}{A^{1-z} B^z}$$

Will satisfy the conditions of the lemma with the constants A, B replaced by 1. All we need to do is thus bound this new function by 1, to get the precise bound on f given in the Lemma. Hence we can now specialize to trying to prove the lemma when $A = B = 1$.

When $A = B = 1$, the main important guiding counterexample to maximum modulus, is the UBA function

$$F(z) = \exp(i \ \exp(-\pi \, i \, z) \ ).$$

The amazing feature of our guiding counterexample

$$F(z) = \exp(i \ \exp(-\pi \, i \, z) \ )$$

Is the way that it anchors on the boundary lines $\{x + iy : x = 0 \text{ or } 1\}$. It is uniformly bounded on those lines, but it is unbounded on the strip, making it by definition an UBA function. Let's check this because it is the crucial point of the discussion. We are playing with the exponential like we did before except that then we were looking at $\exp(cx)$ when c was real, whereas now we are looking at the specific imaginary value $c = -\pi \, i$, and after understanding this function, we are going to need to take an additional exponential to achieve our goals of producing a counterexample.

$$i \ \exp\big(-\pi \, i \, (x + iy)\big) = i \exp(-\pi \, i \, x) \ \exp(\pi \, y).$$

This expands as $\exp(\pi y)(\ \sin(\pi x) \ + \ i \cos(\pi x) \ )$

The reason that we chose the value π is that the imaginary part $\sin(\pi x)$ vanishes on the boundary lines where $x = 0 \text{ or } 1$. It is strictly positive in the strip.

The modulus of the function

$$F(x+iy) = \exp(i \ \exp(-\pi\, i\, (x+iy)) \ )$$

is $\exp\ (\ \sin(\pi x) \ \exp(\pi y) \ )$. At points in the strip with $0 < x < 1$, we see that $\sin(\pi x)$ is positive, and so $F(x+iy)$ has exponential growth to infinity as $y \to \infty$, and exponential decrease to zero as $y \to -\infty$. Most importantly however, when $x = 0 \text{ or } 1$, the modulus of F is just 1. We see how F amazingly anchors the boundary lines. We also see that if we were to replace π in the definition of F by a slightly smaller constant, $\pi - \delta$, for a positive $\delta < \pi$, this would fail. In that case, the function would be exponentially increasing on the boundary lines as well. It would not be UBA. This illustrates the presence of the positive constant δ in the lemma. The function with the coefficient $\pi - \delta$ does satisfy the growth condition in the lemma, but yet it fails to be bounded on the strip. It does not contradict the Lemma, because it fails to be bounded on the boundary lines. The Lemma is telling us that if f is UBA then it must be growing fast along the strip. If f is complex-analytic but slow growing along the strip, the rate $\exp(\exp|cz|)$ with $|c| < \pi$, it must be unbounded on at least one of the boundary lines. Its modulus cannot tend to infinity at a weak rate along the middle of the strip while staying uniformly bounded on the boundary lines. That is the case which is not allowed. It is impossible. Surely we should be able to understand this on an intuitive level. Assume that a complex-analytic function is tending to infinity along a sequence of points with y-values $y_1, y_2, \ldots$ in the strip whilst being uniformly bounded on the boundary lines, $x = 0 \text{ or } 1$. The partial x-derivative of the function must also be big at some points in the strip with y-values $y_1, y_2, \ldots$, because the function has to get back down at the boundary as x varies in the interval from 0 to 1. However, the partial x-derivative has the same modulus as the partial y-derivative from the Cauchy Riemann equations for complex-analytic functions. This forces the derivative of the function to be big in the y-direction as well, which could be driving it to become even larger. However, this real dotty reasoning only seems to hint that exponential increase of the function might be expected. Complex analyticity is a very strong condition. We need a more delicate harmonic analysis to prove the double exponential increase claimed in the lemma.

We remark that there is some argument that instead of proving the Lemma on the complex strip

$\{0 \le \mathrm{Re}(z) \le 1\}$, one should work on the complex strip $\{-1/2 \le \mathrm{Re}(z) \le 1/2\}$. This is purely because in that case the guiding counterexample is

$$F_0(z) = \exp(\exp(\pi\, i\, z)).$$

The only advantage is having one fewer negative sign and appearance of i and so there is slightly less risk of computational dyslexia. The modulus of $F_0(z)$ is just $\exp (\cos(\pi x) \exp(-\pi y))$, which equals 1 on the boundary $x = \pm 1/2$ where the cosine factor vanishes. At points in the strip where $|x| < 1/2$, we see that $\cos(\pi x)$ is positive, and so F has exponential decay as $y \to \infty$, and exponential increase as $y \to -\infty$. We obtain F from F_0 just by rotating and translating the complex variable z, namely $F(z) = F_0(1/2 - z)$. Indeed,

$$\exp\left(\pi \mathrm{i}(1/2 - \mathrm{z})\right) = \exp(\pi i/2) \exp(-\pi\, i\, z)$$
$$= i \exp(-\pi i\, z).$$

Using the symmetric strip $\{-1/2 \le \mathrm{Re}(z) \le 1/2\}$ is appealing but of course many would claim it is the same exact problem. However, we can continue our discussion there for now.

Of course there are many other UBA functions on the strip in addition to F_0. For example, we could replace π in the definition of F_0 by any odd multiple of $k\pi$ with k an odd integer, and that would also give a function which has modulus 1 on the lines with $x = \pm 1/2$. In particular, when we choose $-\pi$, we just get the function $F_0(-z)$, the composition of F with a half turn of z around the origin, which then has exponential growth as $y \to +\infty$, and exponential decrease to zero as $y \to -\infty$, and otherwise has the same properties as F in terms of the upper bounds on the growth of $|F(z)|$ as a function of $|z|$. The other choices where we replace π in the definition of F_0 with an odd multiple $k\pi$ with $|k| > 1$, have larger growth. In fact, we can get UBA functions which have arbitrarily large growth on the interior of the strip as $y \to \infty$, just by composing f with entire functions such as polynomial functions or exponentials. For example,

$$\exp(\ \exp\ (\ \exp(\pi\, i\, z)\ )\ ).$$

The fact that F_0 is bounded on the lines $x = \pm 1/2$ means that any entire function of F_0 will also be. However, whilst we can get functions of enormous growth by composing the function F_0 with more and more exponentials, we cannot in general go the other way and take an UBA function and expect to get another one by taking a logarithm. This is because the logarithm of a function which is bounded on the boundary lines may fail to be bounded on the boundary lines. For example,

$$\log F_0(x+iy) \;=\; \exp(-\pi y)(\;\cos(\pi\, x) + \;i\sin(\pi\, x)\;).$$

Whilst this has zero real part on the lines $x = \pm 1/2$, its imaginary part oscillates unboundedly as $y \to -\infty$ along the boundary lines, albeit remaining pure imaginary. That extra exponentiation is what took this function and produced one which is uniformly bounded on the boundary lines because the exponentiation sends the entire imaginary line to the unit circle which is bounded. Exponentiation takes any vertical straight line and sends it to a bounded circle. There are other functions with lower growth which do this, for example the function $z \to 1/(z+1)$, but such functions are bounded on the whole strip. Although taking the logarithm of an UBA function on the strip, does not necessarily lead to another UBA function on the strip, we can certainly get an UBA function which grows more slowly than any one we care to choose, just by multiplying it by a rational function like $1/(z+1)$, or various decaying exponentials. There is no UBA function of minimal growth is what we strongly suspect from this. We also see other UBA functions growing substantially slower than F_0 for example consider

$$\varepsilon \ \exp(\varepsilon \ \ \exp(\pi \, i \, z) \ \ ).$$

It is still equal to 1 on the boundary lines but yet it modulus tends to ∞ along other lines strip as z tends down the strip in the negative y-direction.

The barrier function proof of the three line's lemma involves finding a clever one parameter family of functions $g_\varepsilon(z)$ on the strip, which tends to 1 pointwise as $\varepsilon \to 0$, but which can be multiplied by the function f to damp it down at infinity enabling the use of the maximum modulus principle on rectangles.

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