Lecture 2. Resolvent of the operator $D^2 + p$ on $\mathbb{R}$.

Lecture Notes for Functional Analysis
December 1, 2025
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Computer Apps: WinEdt, Adobe Acrobat

Acknowledgements: Cambridge University 1983 – 1987.
Prerequisites for this lecture: Real Analysis up to the Fundamental Theorem of Calculus, Multivariable Calculus up to partial differentiation, Complex Analysis up to the Cauchy integral formula for an analytic function and its derivatives.

The celebrated Contraction Mapping Theorem is useful in analysis and topology and we are going to apply it to obtain uniqueness and regularity results for the resolvent.

Contraction Mapping. If (Y, d) is a metric space, then T is called a contraction of Y if T is a mapping from Y to Y, and there exists a constant C with $0 < C < 1$, such that whenever $y, z \in Y$, then

$$d(Ty, Tz) \;\le\; C\, d(y, z).$$

The constant C is called a contraction constant for T. (C is only called a contraction constant if C is less than 1.) The minimum of these constants is called the best contraction constant for T.

A fixed point of T is a point $y \in Y$ with $Ty = y$.

Contraction Mapping Theorem. If Y is a non-empty complete metric space and T is a contraction of Y, then

(a) There exists a unique fixed point of the mapping T.

(b) For every y_0 in Y, the unique fixed point of T can be obtained as a limit

$$\lim_{j\to\infty} T^j y_0.$$

In particular the limit always exists.

(c) For every y_0 in Y, the distance from y_0 to the fixed point y_f of T satisfies

$$d(y_0, y_f) \ \le \ \frac{1}{1-C}\, d(y_0, Ty_0),$$

where C is any contraction constant for T.

Proof. First we note that if a fixed point y exists then it is unique. Indeed, if $Ty = y$ and $Tz = z$ then

$$d(y,z) \ = \ d(Ty,Tz) \ \le \ C\, d(y,z).$$

Thus $(1-C)d(y,z) \le 0$, and because a metric is always non-negative, this implies $d(y,z) = 0$.

To see that a fixed point exists, since Y is non-empty we can select a point y_0 in Y.

Define a sequence of points in Y recursively by $y_{j+1} = Ty_j$. Then for $j \ge 1$,

$$d(y_j, y_{j+1}) = d(Ty_{j-1}, Ty_j) \le Cd(y_{j-1}, y_j),$$

and so by induction, $d(y_j, y_{j+1}) \le C^j d(y_0, Ty_0)$.

If $j < k$, then applying the triangle inequality for the metric we get

$$d(y_j, y_k) \ \le \ d(y_j, y_{j+1}) + d(y_{j+1}, y_{j+2}) + \ldots + d(y_{k-1}, y_k).$$

This is bounded by $(C^j + C^{j+1} + \cdots + C^{k-1}) d(y_0, Ty_0)$ which by summing the geometric series is bounded by a multiple of C^j, explicitly

$$d(y_j, y_k) \;\leq\; \frac{C^j}{1 - C}\, d(y_0, Ty_0). \tag{1}$$

Hence the sequence y_j is Cauchy in Y and since Y is complete y_j converges to some limit $y \in Y$.

To show y is a fixed point of T, it is easy to see that a contraction is continuous and so $y_{j+1} = Ty_j \to Ty$ as $j \to \infty$. By the uniqueness of limits, $Ty = y$. To see this directly,

$$d(y_{j+1}, Ty) = d(Ty_j, Ty) \leq C d(y_j, y) \to 0.$$

To summarize, we started with a completely arbitrary point y_0 in Y, and we showed that the limit in (b) exists and is a fixed point of T. Having showed already that a fixed point is unique, we conclude (a) and (b). To show (c), consider (1) with $j = 0$, and let $k \to \infty$.

Let's introduce the differential equation we hope to solve. If p is a continuous function on the real line and z is a complex number, our goal is to find all the solutions f to the equation

$$\left(-D^2 \;+\; p \;-\; z\right)\, f \;=\; 0. \tag{2}$$

By superficially comparing this to the equation $D^2 f = 0$, which is solved by repeated integration, we might hope that the solution to (2) is unique if we fix a real number t_0, complex numbers a and b, and an "initial" condition

$$f(t_0) \;=\; a, \qquad f'(t_0) = b. \tag{3}$$

The terminology "initial condition" comes from physical problems where $f(t)$ is a function of time.

The problem with the differentiation operator is that it unstable. If we begin with a twice differentiable function and differentiate it twice, we just get a continuous function and cannot necessarily twice differentiate again. Differentiation of functions can lead to very large functions even if we start with functions which are bounded and continuous. In particular tiny oscillations can become big when we differentiate. Surely if we want to analyze differentiation it would seem easier to work with the opposite of differentiation? Integration is the opposite of differentiation and tends to smooth functions out and it certainly takes continuous functions to continuous functions. That is why our first effort will be to change the differential equation (2)-(3) into an integral equation. Heuristically, we write (2) in the form

$$D^2 f = (p - z)f,$$

and use the fundamental theorem of calculus to rewrite it as

$$f = \int\int (p - z)f.$$

Then we will define an operator

$$Tf = \int\int (p - z)f,$$

which we will find is a contraction if we restrict the function $f(t)$ to a small interval $(t_0 - \varepsilon, t_0 + \varepsilon)$. From this we get a solution using the Contraction Mapping Theorem.

We need to be careful about defining the limits of integration in the integrals to make them definite. This will lead us to magically incorporate the initial conditions and the differential equation into the single formula

$$f(t) \;=\; a \,+\, b(t-t_0) \,+\, \int_{t_0}^{t} (t-s)\,(p(s)-z)\,f(s)\,ds. \quad (4)$$

Proposition 1. (Changing a second order linear differential equation with initial conditions into an integral equation.) Suppose p is a continuous function on $\mathbb{R}$, and z is a complex number.
(a) If f is twice continuously differentiable complex function on $\mathbb{R}$ which satisfies (2) and (3), then f satisfies (4).
(b) Conversely, if a continuous function f satisfies (4) then in fact f is twice continuously differentiable and satisfies (2) and (3).

(c) The first derivative of a function f which satisfies (4) or equivalently (2)-(3) is given by

$$f'(t) \;=\; b \,+\, \int_{t_0}^{t} (p(s)-z)\,f(s)\,ds.$$

(d) If p is k-times continuously differentiable, and if f satisfies (4) or equivalently (2)-(3), then writing (2) as

$$f''(t) \;=\; (p-z)f(t).$$

we immediately see that f is $k+2$-times continuously differentiable.

The proof of Proposition 1 is left as an exercise which can be solved with a real analysis course that covers the Fundamental Theorem of Calculus.

We are going to show that we can solve this equation (2)-(3) or equivalently (4) uniquely on the real line for every choice of real t_0 and complex z, a, b. Before carrying out this application of the Contraction Mapping Theorem, let's make some more basic observations about the solutions assuming that they do indeed exist and are indeed unique as we will go on to show.

We notice that the equation (2) is a linear equation. In particular if f and g are solutions then so are $f + g$ and cf where c is any complex number. Let's denote the solution to (2)-(3) or equivalently (4), by $f(z, a, b, t_0,\ t)$. When the initial time t_0 is fixed, let's distinguish two solutions with the initial data $f(t_0) = 1,\ f'(t_0) = 0$, and the initial data $f(t_0) = 0,\ f'(t_0) = 1$, often called Dirichlet and Neumann boundary conditions respectively.

$$\begin{aligned} f_0(z, t_0\ ,\ t) &= f(z, 1, 0, t_0,\ t), \\ f_1(z, t_0\ ,\ t) &= f(z, 0, 1, t_0,\ t). \end{aligned}$$

Then the uniqueness of the solution proves that

$$f(z, a, b, t_0,\ t) \ = \ a\ f_0(z, t_0,\ t)\ +\ b\ f_1(z, t_0,\ t).$$

because by plugging the right hand side into (2) and evaluating at $t = t_0$, we find that it satisfies (2)-(3) or equivalently (4). Hence the behavior of the solution in terms of a and b is very simple. It is not an aspect of the problem we need to analyze analytically. However, studying the dynamics of the solution data $(f(t), f'(t))$ as t varies is an interesting dynamical system in the phase space $\mathbb{R}^3$, which places a copy of $\mathbb{R}^2$ at each point t of the real line. The second order differential equation (2) is a first order differential equation in phase space of the form

$$D_t \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p(t) - z & 0 \end{pmatrix} \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix}.$$

It is common in the study of ordinary differential equations to work with such first order systems and transform them to integral systems. However, we will instead work with the scalar integral equation (4). We already remarked that

$$\begin{pmatrix} f(z, a, b, t_0,\ t) \\ f'(z, a, b, t_0,\ t) \end{pmatrix} = \begin{pmatrix} f_0(z, t_0,\ t) & f_1(z, t_0,\ t) \\ f_0'(z, t_0,\ t) & f_1'(z, t_0,\ t) \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

A fundamental object of study is thus the evolution of the matrix on the right hand side, which obviously satisfies the equation.

$$D_t \begin{pmatrix} f_0 & f_1 \\ f_0' & f_1' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ p(t) - z & 0 \end{pmatrix} \begin{pmatrix} f_0 & f_1 \\ f_0' & f_1' \end{pmatrix},$$

with initial condition

$$\begin{pmatrix} f_0(z, t_0,\ t_0) & f_1(z, t_0,\ t_0) \\ f_0'(z, t_0,\ t_0) & f_1'(z, t_0,\ t_0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The determinant of such a matrix associated to two functions f_0 and f_1 is often known as the Wronskian Determinant. When those functions are solutions to a linear differential equation like (2), their Wronskian miraculously turns out to satisfy a simpler differential equation. In this case it turns out to be constant.

$$W = \begin{vmatrix} f_0 & f_1 \\ f_0' & f_1' \end{vmatrix}.$$

Then

$$W = f_0 f_1' - f_0' f_1,$$

and

$$D_t W \;=\; f_0'\, f_1' \;+\; f_0\, f_1'' \;-\; f_0''\, f_1 \;-\; f_0'\, f_1',$$

The first and last terms cancel and we can use the equation to write the middle two terms as

$$f_0\,(p-z)\,f_1 \;-\; (p-z)\,f_0 f_1 \;=\; 0.$$

Hence the Wronskian determinant is constant. So far what we have done works for any two solutions f_0 and f_1. We have shown in particular that if they provide a basis for the initial conditions of a solution at any point then they continue to do so. The Wronskian can be evaluated by plugging in $t = t_0$ to get in this case,

$$W \;=\; \begin{vmatrix} f_0(z,t_0,\; t_0) & f_1(z,t_0,\; t_0) \\ f_0'(z,t_0,\; t_0) & f_1'(z,t_0,\; t_0) \end{vmatrix} \;=\; \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \;=\; 1.$$

The mapping

$$\begin{pmatrix} f(t_0) \\ f'(t_0) \end{pmatrix} \;\to\; \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix}$$

which takes initial data at t_0, constructs a solution with that data, and then reads off the data at t, is a linear isomorphism of $\mathbb{R}^2$ which preserves area. This can be stated in general terms by saying that the flow of the ordinary differential equation (2) preserves the area form on the phase space. Even more fanciful, is to say that the flow of the ordinary differential equation preserves the symplectic form $dt \wedge d\tau$ on phase space, or $dx \wedge d\xi$ if we have a differential equation in a space variable x. This property continues to hold for the geodesic flow on a Riemannian manifold in higher dimensions, and for other flows associated to many linear ordinary differential equations in higher dimensions.

It is time to show how to apply the beautiful and simple Contraction Mapping Theorem to prove the short time existence of solutions to the homogeneous resolvent equation (4) where z, a, and b are fixed. We will later extend this to long time existence, and investigate the behavior of the solutions as functions of z as well.

Theorem 1. Suppose that p is a continuous function on $\mathbb{R}$. Suppose z, a and b are complex numbers, t_0 is a real number, and ε is a real number such that

$$\sup\{\ |p(t)|\ :\ |t-t_0|<\varepsilon\}\ +\ |z|\ <\ \frac{2}{\varepsilon^2}.$$

This is achievable by the continuity of p at t_0.

Set $\Omega=(t_0-\varepsilon\ ,\ t_0+\varepsilon)$, and let Y be the space of bounded continuous functions on Ω with norm

$$\|f\|\ =\ \|f\|_\Omega\ =\ \sup_\Omega |f|.$$

Then there is a unique solution f to the equation (4) on the interval Ω. In fact, the operator

$$Tf(t)\ = \qquad\qquad a\ +\ b(t-t_0)\ +\ \int_{t_0}^t (t-s)\,(p(s)-z)\,f(s)\,ds \qquad (5)$$

is a contraction on Y with contraction constant

$$\frac{\varepsilon^2}{2}\left(\|p\|_\Omega+|z|\right).$$

Proof of Theorem 1. First note that Y is a normed vector space and hence a metric space. Moreover, Y is complete because if f_j is a Cauchy sequence then it is pointwise Cauchy, and hence converges pointwise to a function. However, the fact that the sequence was uniformly Cauchy easily implies that the convergence is uniform. The limit is thus bounded and continuous being a uniform limit of bounded continuous functions. We leave the real-analysis-1 details for the student.

We need to check that T maps Y to Y. First note that clearly $Tf(t)$ is continuous on Ω. It is also clearly bounded on Ω. Indeed, for $|t - t_0| < \varepsilon$,

$$\left| \int_{t_0}^{t} (t-s)\, ds \right| \;=\; \frac{(t-t_0)^2}{2} \;<\; \frac{\varepsilon^2}{2}.$$

Crude estimates on (5) show that the values of $|Tf|$ are bounded on Ω by

$$|a| \;+\; |b|\varepsilon \;+\; \frac{\varepsilon^2}{2}\,(\|p\|_\Omega + |z|)\,\|f\|.$$

To show that T is a contraction, we have

$$\begin{aligned} |Tf(t) - Tg(t)| \;&=\; \\ &\left| \int_{t_0}^{t} (t-s)\,(p(s)-z)\,(f(s)-g(s))\;ds \right| \\ &\le\; \frac{\varepsilon^2}{2}\,(\|p\|_\Omega + |z|)\quad \|f-g\|. \end{aligned}$$

This shows that T is a contraction by our assumption on ε.

We conclude directly from the Contraction Mapping Theorem that T has a unique fixed point f in Y which is seen from the definition of T to be a solution to (4).

By Proposition 1, the solution f is twice continuously differentiable and solves (2)-(3). Moreover if p is k times continuously differentiable then f is $k+2$ times continuously differentiable.

Our next task is to extend the solution f to (4) to the whole real line. This will be done by covering the real line with intervals Ω corresponding to various different choices of t_0, and solving (4) on these intervals with compatible initial conditions which ensure that the solution fits together continuously and differentiably along the real line. It is important to first establish a general result on the uniqueness of solutions in order that we only have one choice of how to extend a solution.

Theorem 2. Suppose that twice differentiable functions f and g satisfy the equation (2) on open intervals I and J respectively. Suppose that there exists some point $t_1 \in I \cap J$ such that

$$f(t_1) = g(t_1), \qquad f'(t_1) = g'(t_1).$$

Then $f = g$ on $I \cap J$.

Proof. Suppose on the contrary, that the solutions f and g do not agree everywhere on the interval $I \cap J$. Let S be the set of points t in $I \cap J$, with $t < t_1$ and $f(t) \neq g(t)$. If S is non-empty. Then it has a supremum t_0 in $I \cap J$ with $t_0 \leq t_1$. We claim that

$$f(t_0) = g(t_0), \qquad f'(t_0) = g'(t_0).$$

Indeed, if $t_0 = t_1$ then this holds by assumption, and if $t_0 < t_1$, then it holds by continuity because $f = g$ on $(t_0, t_1]$.

Now by Theorem 1, we can pick $\varepsilon > 0$ such that a solution to (4) with initial data $a = g(t_0)$ and $b = g'(t_0)$ exists and is unique. We can moreover shrink ε if necessary so that the interval $\Omega = (t_0 - \varepsilon\ ,\ t_0 + \varepsilon)$ is contained in $I \cap J$. Now by restricting the solutions f and g to Ω, we find that they both agree with the solution from Theorem 1, and hence they agree with each other on a neighborhood of t_0, contradicting t_0 being the supremum of S. Hence S is empty. A similar argument shows that the set of points in $I \cap J$ which are greater than t_1, and where f and g differ is empty, and so $f = g$ on $I \cap J$.

Corollary 2.1. There is at most one solution to (2)-(3) on any interval containing t_0, (including the whole of $\mathbb{R}$).

Proof. This follows immediately from Theorem 2, by taking I and J to be the interval in question and $t_1 = t_0$.

Theorem 3. There is a unique solution to (2)-(3) equivalently (4), on the whole real line.

Proof. It is enough to show that for every $N > 0$, there there exists a unique solution to (2)-(3) on $(t_0 - N\ ,\ t_0 + N)$. This proves Theorem 3, for if we want to get a solution f on the whole of $\mathbb{R}$, to define $f(t)$, we simply select any $N > |t - t_0|$ and define $f(t) = f_N(t)$ where $f_N(t)$ is the solution of (2)-(3) on $(t_0 - N\ ,\ t_0 + N)$. This is well defined, because it does not depend on the choice of N by Theorem 2. This definition satisfies the conditions (3) at t_0, and satisfies the differential equation (2) at t, which is a local equation only depending on the values of f on an arbitrarily small neighborhood of t.

At the moment we can only get the existence of solutions to (2) with initial conditions (3), on an ε neighborhood of the fixed point where we specify the initial conditions. We are going to need to sew together many such solutions, and we will later select points t_j indexed by integers j with $0 < |j| \leq J$, and solve (2) on each of the intervals $(t_j - \varepsilon \ , \ t_j + \varepsilon)$ using Theorem 1. What we need to accomplish this is firstly to get an $\varepsilon > 0$ which satisfies the conditions of Theorem 1 at all the points t_j. This is the reason we restricted our attention to a bounded interval $(t_0 - N \ , \ t_0 + N)$. Indeed, the continuous function $p(t)$ on $\mathbb{R}$ is uniformly bounded on each bounded subset of $\mathbb{R}$. Choose ε with $0 < \varepsilon < 5$ to satisfy

$$\sup\{ \ |p(t)| \ : \ |t - t_0| < N + 5\} \ + \ |z| \ < \ \frac{2}{\varepsilon^2}.$$

There is nothing special about the value 5 in defining ε. It could be replaced with ε in the condition just above, if we desired to optimize ε at this point. On the other hand, we could have limited it by 5 in this way in Theorem 1 in order to simplify the consideration, hardly altering the result, but we decided instead to use the slightly sharper condition since the size of interval on which (5) is a contraction could be of interest for some applications. Now if we have a point $t_j \in [t_0 - N \ , \ t_0 + N]$, then

$$\frac{\varepsilon^2}{2} \ (\sup\{|p(t)| : |t - t_j| < \varepsilon\} \ + |z|) \ < \ 1.$$

This means that we can apply Theorem 1 to solve (2) on $(t_j - \varepsilon \ , \ t_j + \varepsilon)$ for any choice of $t_j \in [t_0 - N \ , \ t_0 + N]$, with any initial data we care to choose at t_j of the form

$$f(t_j) \;=\; a_j, \qquad\qquad f'(t_j) \;=\; b_j.$$

We now want to specify the values of a_j and b_j so that the solutions we obtain this way all join up to form a continuous and differentiable function on $[t_0 - N \;,\; t_0 + N]$. We want to be able to move from one interval to the next, and so we want t_j to be in the preceding interval. This means we simply want $0 < t_j - t_{j-1} < \varepsilon$, in order to go forwards through positive values of the index j, and have $t_j \in (t_{j-1} \;,\; t_{j-1} + \varepsilon)$, to go backwards as we increase the magnitude of negative values of the index j, and have $t_j \in (t_{j+1} - \varepsilon \;,\; t_{j+1})$. We can accomplish this just by for example setting

$$t_j \;=\; t_0 \;+\; j\varepsilon/2, \qquad\qquad |j| \leq 2N/\varepsilon.$$

It is easy to check that the interval $(t_0 - N \;,\; t_0 + N)$ is contained in the union of the intervals $(t_j - \varepsilon \;,\; t_j + \varepsilon)$.

To construct a solution F to (2)-(3) on $(t_0 - N \;,\; t_0 + N)$, we first solve (2)-(3) to get a function $F(t)$ on $(t_0 - \varepsilon \;,\; t_0 + \varepsilon)$. Next we solve (2) on $(t_1 - \varepsilon \;,\; t_1 + \varepsilon)$ with the initial data

$$f(t_1) = F(t_1), \qquad\qquad f'(t_1) \;=\; F'(t_1).$$

The solutions f must agree with F on the intersection by Theorem 2. This enables us to extend F to the union $(t_0 - \varepsilon \;,\; t_1 + \varepsilon)$ simply by defining F to equal f at the new points in its domain. By induction we continue. Once F is defined on $(t_0 - \varepsilon \;,\; t_{j-1} + \varepsilon)$, we extend it to $(t_0 - \varepsilon \;,\; t_j + \varepsilon)$ by solving (2) to get a new function f on $(t_j - \varepsilon \;,\; t_j + \varepsilon)$, using the initial data

$$f(t_j) = F(t_j), \qquad\qquad f'(t_j) \ = \ F'(t_j).$$

Eventually we will have constructed F over $(t_0 - \varepsilon \ , \ t_0 + N)$. We then work backwards in the same fashion, solving the equation (2) on $(t_j - \varepsilon \ , \ t_j + \varepsilon)$ with the same initial conditions as F at t_j which are given above, to extend the solution from $(t_{j-1} - \varepsilon \ , \ t_0 + N)$ to $(t_j - \varepsilon \ , \ t_0 + N)$ as j decreases from -1 to $-J$, eventually obtaining a solution F on $(t_0 - N \ , \ t_0 + N)$. We can alternatively alternate and extend the solution to the intervals $(t_j - \varepsilon \ , \ t_j + \varepsilon)$ and $(t_{-j} - \varepsilon \ , \ t_{-j} + \varepsilon)$ at each step of the induction.

If the function $p(t)$ is uniformly bounded on $\mathbb{R}$, we could continue this to define F on the whole real line. However, if $p(t)$ is not uniformly bounded on $\mathbb{R}$ and we want to construct the solution on the whole real line we might need to start reducing the size ε of the intervals on which we extend. We will be able to achieve the extension in countably many consecutive steps which is left as an exercise for the reader.

Theorem 4. Suppose that p is a continuous function on $\mathbb{R}$. Suppose a and b are complex numbers and t_0 is a real number. For each $z \in \mathbb{C}$, suppose $f(z,t)$ is the continuous solution to (2)-(3) or equivalently (4) in the variable t, for $t \in \mathbb{R}$. Then $f(z,t)$ is jointly continuous in (z,t) and analytic in z. Moreover the partial derivatives

$$\frac{\partial^j D_t^k f}{\partial z^j}(z,t)$$

are all jointly continuous in (z,t), where j is a counting number, and k is a counting number with $k \leq K+2$, where

$p(t)$ is K times differentiable on $\mathbb{R}$. If p is real-analytic in t then so is $f(z,t)$.

Proof. When t is close to t_0, the proof of the continuity of $f(z,t)$ is virtually indistinguishable from the proof of Theorem 1, except for keeping track of z and its effects on the constants. It really may not be necessary to set up the proof of Theorem 1 all over again to do this, but it saves pondering.

For $R > 0$, let B_R be the disc of complex numbers z with $|z| < R$. By the continuity of p, we can select $\varepsilon > 0$ such that such that

$$\sup\{\ |p(t)|\ :\ |t - t_0| < \varepsilon\}\ +\ R\ <\ \frac{2}{\varepsilon^2}.$$

The only difference with Theorem 1, is that we have replaced $|z|$ by R, so that this ε will produce a contraction with uniform contraction constant for $|z| < R$.

Set $\Omega = (t_0 - \varepsilon\ ,\ t_0 + \varepsilon)$, and

$$\|p\|_\Omega = \sup\{\ |p(t)|\ :\ |t - t_0| < \varepsilon\}.$$

We let Y denote the space of bounded complex valued continuous functions f on $B_R \times \Omega$ with the sup norm, which is a complete metric space. For a function f in Y, we define

$$Tf(z,t)\ =$$

$$a\ +\ b(t - t_0)\ +\ \int_{t_0}^{t} (t - s)\,(p(s) - z)\,f(z,s)\,ds.$$

This formula is jointly continuous in (z,t). Moreover, $|Tf(z,t)|$

is bounded by

$$|a| \ + \ |b|\varepsilon \ + \ \frac{\varepsilon^2}{2}\left(\|p\|_\Omega + R\right)\|f\|.$$

Hence T maps Y to Y. To show that T is a contraction, $|Tf(z,t) - Tg(z,t)| \ =$

$$\left|\int_{t_0}^{t}(t-s)\left(p(s)-z\right)\left(f(z,s)-g(z,s)\right)\,ds\right|$$

$$\leq \ \frac{\varepsilon^2}{2}\left(\|p\|_\Omega + R\right) \quad \|f-g\|.$$

We conclude directly from the Contraction Mapping Theorem that T has a unique fixed point f in Y which is jointly continuous in the variables (z,t). For z held constant, we recognize that fixed points of T are the solutions of (4), equivalently (2)-(3) for $t \in \Omega$.

We also note from Proposition 1, that the first derivative of the solution f with respect to t is

$$f'(z,t) \ = \ b \ + \ \int_{t_0}^{t}\left(p(s)-z\right)f(z,s)\,ds.$$

It is an exercise in real-analysis-1 to show that this is continuous on $B_R \times (t_0 - \varepsilon \ , \ t_0 + \varepsilon)$, whenever $f(z,t)$ is continuous on $B_R \times (t_0 - \varepsilon \ , \ t_0 + \varepsilon)$.

The fact that higher t-derivatives of the solution f are continuous in (z,t) follows from the differential equation

$$(D_t^2 f)(z,t) \ = \ (p(t) - z) \ f(z,t).$$

If we know p is k-times continuously differentiable, by induction we find that for $j \geq 2$,

$$(D_t^j f)(z,t) \ = \ u_j(z,t)f(z,t) \ + \ v_j(z,t)f'(z,t),$$

where $u_j(z,t)$ and $v_j(z)$ are polynomials of degree $j-1$ and $j-2$ in z respectively, and the coefficients of these polynomials are $k-j$-times and $k-j+1$ times continuously differentiable in t respectively. Theorem 4 thus follows if we can show that $f(z,t)$ is analytic in z and the z-derivatives are continuous in (z,t), since the same follows for the first t-derivative and the higher t-derivatives of f from these expressions.

Part (b) of The Contraction Mapping Theorem tells us that the fixed point of the Contraction is the limit of the iterates $T^j f_0$ for any choice of $f_0 \in Y$. In particular, in the current case, we can start the approximation with the function $a+b(t-t_0)$ and we find that the solution to (2)-(3) equivalently (4) is

$$\lim_{j\to\infty} T^j\left(a+b(t-t_0)\right).$$

From the formula for T, we see that the jth iterate is a polynomial in z of degree j.

Theorem. A uniformly convergent limit of analytic functions defined on an open subset of the complex plane is analytic.

Proving this is an exercise in complex analysis, for example using the Cauchy integral formula which represents an analytic function inside a disc as a weighted integral of the function around the boundary circle. It turns out the pleasant bounded integration in the Cauchy integral formula can easily be switched with taking a uniform limit of analytic functions, and is also uniform in the variable t.

We have thus proven that the solution $f(z,t)$ is analytic in z on $B_R \times \Omega$. If $p(t)$ is real analytic and we start with $f_0(t) = a + b(t - t_0)$, we see that the iterates of $T^j f_0$ are also analytic in t, and so the solution being a uniform limit of analytic functions is also analytic in t.

Similarly, expressing the z derivatives of the solution as a Cauchy integral formula around a fixed circle enables us to see that these derivatives are jointly continuous in (z,t).

So far, we have proven Theorem 5 on $B_R \times \Omega$ where $\Omega = (t_0 - \varepsilon\ ,\ t_0 + \varepsilon)$. Now we want to extend this to the whole of $\mathbb{C} \times \mathbb{R}$. For this we return to our general considerations of the solution $f(z, a, b, t_0,\ t)$ which solves (2)-(3), equivalently (4).

Recall we showed in Theorem 2, that a solution $F(z, a, b, t_0,\ t)$ to the differential equation (2)-(3), equivalently (4) exists and is unique for all complex numbers a, b, z and initial point t_0. This solution is unique when its value and derivative is specified at any point. Setting

$$a_1 \ = \ f(z, a, b, t_0,\ t_1), \qquad\qquad b_1 \ = \ f'(z, a, b, t_0,\ t_1),$$

we have

$$f(z, a, b, t_0,\ t_1 + t) \ = \ f(z,\ a_1\ ,\ b_1\ , t_1,\ \ t).$$

We can verify this formula by differentiating both sides with respect to t and noticing they satisfy (2) in the variable t. By Theorem 2, we just therefore need to check that both sides have the same value and derivative at one point, in this case $t = 0$. Setting $t = 0$, we find that the two sides of the formula agree and both equal a_1. Differentiating each side and setting $t = 0$ both become are b_1. Therefore they are equal. However, the right hand side is continuous in (z, t_1), and analytic in z. Hence so is the left hand side, and thus $(z, a, b, t_0,\ t)$ is continuous at $t = t_1$, and analytic in z. We leave it to the reader to collect these details together to verify that we have now shown that Theorem 4 holds on $B_R \times \mathbb{R}$, and since R was arbitrary, it holds on $\mathbb{C} \times \mathbb{R}$.

Example: The case $p = 0$, and $t_0 = 0$. If we solve the equation

$$\left(-D^2\ -\ z\right)\ f\ =\ 0, \qquad f(0) = 1,\ \ f'(0) = 0,$$

We factorize

$$(iD + \sqrt{z})(iD - \sqrt{z})f\ =\ 0,$$

However, writing $w = \sqrt{z}$, we have

$$e^{iwt} iD e^{-iwt}\ =\ iD\ +\ w,$$

so the equation becomes

$$e^{iwt} iD e^{-iwt} e^{-iwt} iD e^{iwt}\ f\ =\ 0,$$

and so

$$D e^{-2iwt} iD e^{iwt}\ f\ =\ 0,$$

and

$$e^{-2iwt} iD e^{iwt}\ f\ =\ C_0(w),$$

$$iD e^{iwt}\ f\ =\ C_0(w)\ \exp(2iwt)$$

and

$$D e^{iwt}\ f\ =\ C_1(w)\ \exp(2iwt)$$

$$e^{iwt}\, f \;=\; C_1(w) \int e^{2iwt}\, dt \;=\; C_1(w)\, \frac{\exp(2iwt)}{2iw} \;+\; C_2(w).$$

Thus

$$f \;=\; \exp(-iwt) \;\left(C_1(w)\, \frac{\exp(iwt)}{2iw} \;+\; C_2(w) \right).$$

However, we don't like this apparent singularity at $w = 0$ because we just proved the function is continuous in (z, t) and so needs to remain bounded when $z = w = 0$. If we think about adjusting the constant to eliminate the pole, the right hand side becomes

$$\exp(-iwt) \;\left(C_1(w)\, \frac{\exp(2iwt) - 1}{2iw} \;+\; C_3(w) \right),$$

$$= \left(C_1(w)\, \frac{\exp(iwt) - \exp(-iwt)}{2iw} \;+\; C_3(w) \exp(-iwt) \right).$$

$$= \;\; C_1(w)\, \frac{\sin wt}{w} \;+\; C_3(w) \exp(-iwt).$$

The power series expansion looks like

$$C_1(w)(t \;+\; O(t^3)) \;+\; C_3(w)(1 \;-\; iwt \;+\; \ldots).$$

Putting in the initial condition that this should look like

$$a \;+\; bt \;+\; O(t^2),$$

we get

$$c_3(w) \;=\; a, \qquad C_1(w) \;-\; iwC_3(w) \;=\; b,$$

which leads to the solution

$$a\;\exp(-iwt) \;+\; (b + iwa)\frac{\sin wt}{w}.$$

However, from our existence result, this is supposed to be analytic in w^2 and should therefore only involve even powers of w. We want to take a look at the function with coefficient a. This is

$$a\left(\exp(-iwt) \;+\; iw\frac{\sin wt}{w}\right) \;=\; a\left(\exp(-iwt) \;+\; i\sin wt\right).$$

Indeed,

$$\exp(-iwt) \;+\; \frac{\exp(iwt) \;-\; \exp(-iwt)}{2} \;=\; \cos(wt).$$

Altogether we get

$$f(w^2, a, b, t) \;=\; a\cos wt \;+\; b\frac{\sin wt}{w},$$

which is indeed analytic in $z = w^2$, requiring no branch cut to be well defined.