Resolvent of the operator $D^2 + p$ on $\mathbb{R}$.

Lecture Notes for Functional Analysis
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Kate Okikiolu

Computer Support by Hans Lindblad
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The functions we consider will be complex valued functions on the real line. Denote by D the operation of differentiation which sends continuously differentiable functions to continuous functions. Let p be a continuous real valued function on $\mathbb{R}$ which has compact support in the interval $(0, N)$. We are interested in understanding the space of functions f which solve the eigenfunction equation

$$-(D^2 f)(t) \ + \ p(t)f(t) \ = \ zf(t), \tag{1}$$

for all $t \in \mathbb{R}$, where z is a complex constant. The equation can be written as

$$\Big(-D^2 \ + \ p \ - \ z\,\Big)\,f \ = \ 0.$$

The current goal of studying these solutions f is to be able to later write down formulas for the resolvent operator

$$\frac{1}{(-D^2 \ + \ p) \ - z}$$

where here z denotes multiplication by z. We are aiming to compute the spectral measure of the opeartor $-D^2 + p$.

Exercise 0. Show that if f_1 and f_2 are solutions to (1) then so are $f_1 + f_2$ and cf_1, where c is any complex number. The space of solutions is thus a vector space.

To understand solutions to (1), we begin by explicitly solving (1) in intervals where p vanishes. In these intervals

$$-(D^2 f)(t) \;=\; zf(t),$$

which can be written in the form $(D^2 + z)f = 0$.

Exercise 1. Factorize the operator $D^2 + z$ and use an integrating factor and the fundamental theorem of calculus to show that every twice continuously differentiable function f with $(D^2 + z)f = 0$ has the form

$$A\, e^{it\sqrt{z}} \;+\; Be^{-it\sqrt{z}}, \tag{2}$$

where A and B are complex constants.

Since p is supported on $(0, N)$, we get from Exercise 1, that any solution f to (1) on the whole real line, will have the form (2) for t in the interval $(-\infty, 0]$, and it will also have the form (2) for t in the interval $[N, \infty)$, although the pair of coefficients A and B might change.

Our next step is to establish the existence of solutions f on the real line by using the Picard iteration method. Let's take a short detour to mention that the general theory of Picard integration starts off with a first order differential equation. It is easy to change a second order equation into a first order vector equation by rewriting (1) as

$$D \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix} \;=\; \begin{pmatrix} 0 & 1 \\ p(t) - z & 0 \end{pmatrix} \begin{pmatrix} f(t) \\ f'(t) \end{pmatrix}$$

Integrating this between $t = t_0$ and a general value of t

changes this into the vector valued integral equation

$$\begin{pmatrix} f(t) \\ f'(t) \end{pmatrix} = \begin{pmatrix} f(t_0) \\ f'(t_0) \end{pmatrix} + \int_{s=t_0}^{t} \begin{pmatrix} 0 & 1 \\ p(s) - z & 0 \end{pmatrix} \begin{pmatrix} f(s) \\ f'(s) \end{pmatrix} ds$$

And the Picard method solves general vector valued first order differential equations. However, we need not work with this vector valued integral equation because we can work more easily with the equivalent equation which writes f as a double integral of its second derivative. Indeed, two applications of the fundamental theorem of calculus gives us

$$f(t) = f(t_0) + \int_{s=t_0}^{t} f'(s)\,ds,$$

$$f'(s) = f'(t_0) + \int_{r=t_0}^{s} f''(r)\,dr.$$

Then substituting $f'' = (p - z)f$ and plugging in gives

$$f(t) = f(t_0) + \int_{s=t_0}^{t} \left(f'(t_0) + \int_{r}^{s} (p(s) - z)\, f(r)\,dr \right) ds$$

After changing the order of integration this becomes

$$f(t) = f(t_0) + (t - t_0) f'(t_0) + \int_{t_0}^{t} (t - r)\,(p(r) - z)\, f(r)\,dr.$$

If we want to find a solution to (1) with the initial conditions

$$f(t_0) = a, \qquad f'(t_0) = b, \tag{3}$$

then we can check this is equivalent to finding a function f satisfying the integral equation

$$f(t) = a + b(t - t_0) + \int_{t_0}^{t} (t - s)\,(p(s) - z)\, f(s)\,ds. \tag{4}$$

It is pleasing how the differential equation (1) plus the initial conditions (3) get incorporated into a single integral equation (4). If this equation fails however and f is just an approximation to a solution, we can hope to find a better approximation Tf which is given by the right hand side. To fix notation, call the right hand side Tf, that is for any continuous function on $\mathbb{R}$, define a new function Tf on $\mathbb{R}$ by

$$Tf(t) \;=\; a \;+\; b(t-t_0) \;+\; \int_{t_0}^{t} (t-s)\,(p(s)-z)\,f(s)\,ds.$$

Exercise 2. If f is a continuous function on $\mathbb{R}$, then Tf is a twice continuously differentiable function, and Tf satisfies the initial conditions (3).

Now if f and g are continuous functions, let's estimate the difference $Tf - Tg$ in terms of $f-g$. We have

$$Tf(t) \;-\; Tg(t) \;=\; \int_{t=0}^{t} (t-s)\,(p(s)-z)\,(f(s)-g(s))\,ds.$$

Exercise 3. For $\varepsilon > 0$, write $J = J_\varepsilon$ for the interval of the real line with center t_0 and radius ε, that is

$$J = (t_0 - \varepsilon \;,\; t_0 + \varepsilon) \;=\; \{\, t \;:\; t_0 - \varepsilon \;<\; t \;<\; t_0 + \varepsilon \}.$$

Then

$$\sup_J |Tf(t) - Tg(t)| \;\leq\; C\,\varepsilon\; \sup_J |f(s) - g(s)|,$$

where

$$C \;=\; \frac{\sup_{\mathbb{R}} |p(t)| \;+\; |z|}{2}.$$

In particular we see that when $C\varepsilon < 1$, or rather to be more constructive, if we choose ε with $0 < \varepsilon < 1/C$, then the operator T is a contraction of the space of continuous functions on J. More correctly we ought to say that the operator T can be restricted to the space S of continuous functions f on J with $f(t_0) = a$, which is a metric space when it is equipped with the supremum norm, and this restriction of T to S is a contraction of S.

Exercise 4. We say that a continuous function f on the interval J is a fixed point of T if $Tf = f$. Such a function f will satisfy the initial conditions (3) and the differential equation (1) on the interval J. Starting with any continuous function f_0 on J, the sequence of functions f_j defined recursively by $f_{j+1} = Tf_j$ will converge uniformly on J to a continuous function f on J which is a fixed point of T. In fact f is the unique solution to (1), (3) on the interval J.

If we start with a solution (2) to (1) on $(-\infty, t_0)$, we have shown that we can extend it to the interval $(-\infty,\ t_0 + \varepsilon)$ where $\varepsilon = 1/(\|p\|_\infty + |z|)$. However, we can iterate this to extend the solution to the whole real line. When we get past $t = N$, we know the solution settles down to have the form (2) from then on with new values of the constants A and B.

Exercise 5. Consider the solution to (1) which equals $e^{it\sqrt{z}}$ for $t < 0$, so that $f(0) = 1$ and $f'(0) = i\sqrt{z}$. Show that this function $f(t) = f_{(z)}(t)$ varies continuously in z.